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AUTHOR(S): W. C. Overton, Jr.

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LOS ALAMOS SCIENTIFIC LABORATORY
Post Office Box 1663 Los Alamos, New Mexico 87545
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ANALYSIS OF THE MACROSCOPIC EQUATIONS FOR SECOND SOUND IN SOLIDS*

W. C. Overton, Jr.
Los Alamos Scientific Laboratory
Los Alamos, N.M. 87545

The microscopic theories of second sound in solids can be expressed in macroscopic form (Guyer and Krumhansl, 1966; Ackerman and Guyer, 1968) when the normal and resistive process phonon relaxation times $\tau_N(\omega,T)$ and $\tau_R(\omega,T)$ have been averaged appropriately over all ω's. Thus, for given temperature T, $\tau_N(T)$ and $\tau_R(T)$ can be regarded as fixed parameters. The macroscopic equations given in Ref. 2 for the heat current in 3-dimensions can be expressed in one dimension according to

$$\left(1 + \tau_o \frac{\partial}{\partial t}\right) \frac{\partial^2 Q}{\partial x^2} - \frac{1}{v^2 \tau_R} \frac{\partial Q}{\partial t} - \frac{1}{v^2} \frac{\partial^2 Q}{\partial t^2} = - F_Q(x_o,\omega)\, \delta(x-x_o)\, e^{-i\omega t}, \quad (1)$$

where $v = c_D/\sqrt{3}$ = second sound velocity, c_D = mean Debye velocity, $\tau_o = 9\tau_N/5$, and we have here inserted an impulse driving function over the plane $x = x_o$. Eq. (1) represents the experimental situation in which the single crystal solid is excited by a heat pulse applied on one face at $x_o = 0$. Both the face at $x = 0$ and the opposite face at $x = L$ serve as reflectors of the heat current Q. Sensitive thermometers on the plane at $x = L$ measure the second sound phenomenon.

Our purpose is to show a method for solving (1) in closed form and for calculating the temperature excursion $\Delta T(L,t)$ for comparison with experiment. Letting $Q(x,t) = \Psi(x,\omega)\exp(-i\omega t)$, we obtain

$$d^2\Psi/dx^2 + k^2\Psi = - F_Q(x_o,\omega)\, \delta(x-x_o)/(1 - i\omega\tau_o) \quad , \quad (2)$$

$$k^2 = (\omega^2 + i\omega/\tau_R)/[\, v^2(1 - i\omega\tau_o)] \, . \quad (3)$$

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In (1), (2) the driving force is confined to the plane $x=x_o$ by the delta function, but this is but a special case of the more general function $F(\xi,\omega)$; $0<\xi<L$. For the more general case the particular integral is

$$\Psi_p = -[k(1 - i\omega\tau_o)]\int_0^x F(\xi,\omega)\,\sin k(x-\xi)\,d\xi \quad , \qquad (4)$$

and the general solution is $\Psi = A \sin kx + B \cos kx + \Psi_p$. Using boundary conditions for total reflection of the heat current at $x = 0$, $x = L$, we obtain $B = 0$ and, using (4), we can evaluate A. The solution can be put in the final form,

$$\Psi(x,\omega) = [k(1 - i\omega\tau_o)]^{-1}\Big[\int_x^L d\xi\, F(\xi,\omega)\sin k(L-\xi)\,\sin kx/\sin kL + \int_0^x d\xi\, F(\xi,\omega)\,\sin k(L-x)\,\sin k\xi/\sin kL\Big] \, . \qquad (5)$$

The response to a forcing function of unit amplitude and frequency $\omega/2\pi$, concentrated at x_o, is then obtained from (5) as the Green's function $\Psi(x,\omega)/F(x_o,\omega)$, in the form

$$G(x,x_o,\omega) = \begin{cases} \dfrac{\sin k(L-x)(1-\cos kx_o)}{k^2(1 - i\omega\tau_o)\sin kL} \; ; & 0< x< x_o \; , \\[2ex] \dfrac{\sin kx[1 - \cos k(L-x_o)]}{k^2(1 - i\omega\tau_o)\sin kL} \; ; & x_o< x< L \; . \end{cases} \qquad (6)$$

A sharp heat pulse applied at $x_o = 0$ can be represented by $f(t) = (Ka/2)\exp(-|t|a)$, the Fourier transform of which is $F(0,\omega)=(Ka^2)/(\omega^2+ a^2)$. Thus, we obtain

$$Q(x,t) = (K/2\pi)\int_{-\infty}^{\infty} d\omega\, G(x,x_o,\omega)\,\exp(-i\omega t)/(1 + \omega^2/a^2) \; . \qquad (7)$$

This line integral can be evaluated for $t \geq 0$ by summing residues of the poles of the $-i\omega$ half plane. This number is finite because the number of allowed values of k is finite, although large. The poles at $-i/\tau_R$, i/τ_o, and 0 have zero residues.

For the plane $x = L$, where thermometers of the experiment are located, the poles associated with $kL = n\pi$ $(n = \pm1, \pm2,\ldots,N_{max})$ are determined by solving the quadratic

$$k^2L^2 = (L^2/v^2)(\omega^2+ i\omega/\tau_R)/(1 - i\omega\tau_o) = n^2\pi^2 \; . \qquad (8)$$

We find

$$\omega_n = \pm\Omega_n - i\zeta_n/2 \; ; \quad \Omega_n = (n^2\pi^2v^2/L^2 - \zeta_n^2/4)^{\frac{1}{2}} \; ; \qquad (9)$$
$$\zeta_n = n^2\pi^2v^2\tau_o/L^2 + 1/\tau_R \; ; \; W_n = i\Omega_n \; ; \; \Omega_n \text{ real for } -N<n<N \; .$$

A second set falls on the $-i\omega$ axis at $\omega_n = \pm iW_n - i\zeta_n/2$ for all $n > N$. In view of (8), and since n occurs only as n^2 in (9), no two poles ever fall at the same point. Thus, it is only necessary to sum residues on the first Riemann sheet. In order to determine N, let $\pm\eta$ be the solution of the quatratic obtained from (9) by setting $\Omega_n = 0$, then N is in the range $\eta - 1 < N < \eta + 1$. Using τ_N and τ_R for solid ^3He at $T < 1$ K, we find N varies from 4 to about 20 and M usually = 1. With $x_o = 0$ in (7) we can obtain the residues and determine $Q(x,t)$. The derivative $\partial Q/\partial x$ then gives $C_v \partial T/\partial t$, which, when evaluated at $x = L$, can be expressed in the

$$C_v \Delta T(L,t) = KL^3 \{ \sum_{n=1}^{N} (n^4 \pi^4 D_n v^2)^{-1} (1-\cos\pi n)[-\Omega_n^2 \cos\Omega_n t - (\Omega_n \zeta_n/2)\sin\Omega_n t] \times e^{-\zeta_n t/2} + \sum_{N+1} (n^4 D_n \pi^4 v^2)(1-\cos\pi n)[W_n^2 \cosh W_n t + (W\zeta/2)\sinh W_n t] e^{-\zeta_n t/2} \} \quad (10)$$

$$D_n = 1 - (n\pi v\tau_o/L + L/n\pi v\tau_R)^2 \ . \quad (11)$$

Instead of using τ_N, τ_R values obtained from theory, we select chosen arbitrary parameters to calculate the waveforms described by (10). The computational procedure is to adjust values until agreement with the experimental second sound waveshape is obtained. The adjusted τ_N, τ_R are then used separately to calculate the thermal conductivity which is then compared with static experimental data. Although the agreement is not quantitative, the result obtained for solid ^3He at 0.4 K is 60% of the static value. Thus, the two approaches are at least qualitatively consistent.

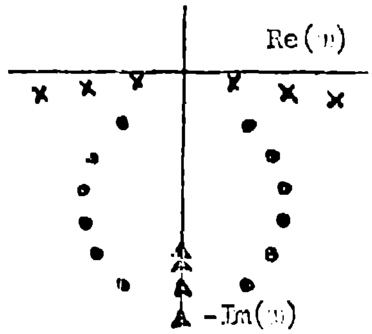
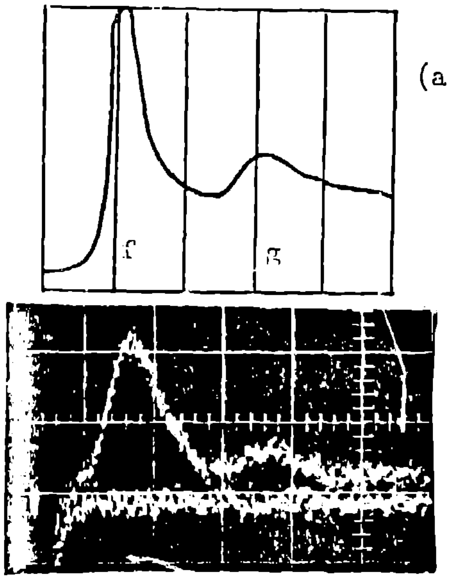


Fig. 1. ● - poles for M< n< N; ▲ - for n > N; ✕ - poles for incorrect transmission line analog of Ref. 2.



(a)

(b)

Fig. 2. (a) Calculated waveform with $\tau_N = 0.09$ μs, $\tau_R = 4.4$ μs. (b) Observed for solid ^3He at 0.5 K; f- first arrival; g- first reflection; 10 μs per division. Ref. 3.

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3. C.C. Ackerman and W.C. Overton, Phys. Rev. Letters 22(1969)764.